a bit of combinatorics

How many positive subsums of a positive sum do we have ?

par Miklós Dezsö

[NDLR : Miklós Dezsö est hongrois ; il a donné sa conférence à l'Arbresle, en anglais, avec traduction simultanée de Pierre Duchet ; le texte de l'auteur est en anglais ; les commentaires insérés en français sont de Pierre Duchet.]

Cet article est typique de la démarche et des raisonnements combinatoires, particulièrement populaires en Hongrie où les mathématiciens y excellent. Les ingrédients de base sont d'une simplicité déconcertante : des éléments (ici des nombres entiers), des paires d'éléments, des triplets, … des ensembles d'éléments ; puis, en compliquant un peu, des ensembles de paires, de triplets, des ensembles d'ensembles …

Une opération élémentaire, l'addition, est examinée sous l'angle combinatoire, c'est-à-dire par les combinaisons, les groupements ("groups" en anglais) qu'elle peut engendrer.

Les ensembles dont il est question ici permettent les répétitions des éléments qui les composent : nous pourrions traduire par "multiensemble" ou par "famille", mais nous garderons les termes de l'auteur : {-1, -2, -3, 6}, {-1, -1, -1, 1, 1, 2} seront donc des ensembles ("sets") à 4 et à 6 éléments.

Ces répétitions sont évidemment permises dans les parties, ou sous-ensembles ("subsets"). Ainsi l'ensemble {-1, -1, -1, 1, 1, 2} contient les parties {-1, 1, 2} et {-1, 1} mais aussi {1, 1, 2}, {-1, -1} et {-1, -1, 1, 1}.

Un ensemble à deux éléments est une ***paire*** *("pair") ; un ensemble à trois éléments est un* ***triple*** *("triple").*

Bien que les éléments considérés ici soient des nombres entiers (positifs, négatifs, ou nuls), les méthodes proposées s'appliquent sans changement à n'importe quels nombres réels, preuve du caractère essentiellement "combinatoire" du problème étudié : il s'agit en effet de chercher des ensembles "positifs", c'est-à-dire des ensembles de nombres dont la somme est positive.

Ainsi
A = {-100, +200}, *B* = {-1, 1, 2}, *C* = {-1, -1, -1, 4} et *D* = {1, 1, 1, -2} sont des ensembles positifs mais {-1, -1}, {-1, -2, -3, 6}, {-1, -1, -1, 3} et {1, 1, 1, -3} ne le sont pas.
• *A* contient (strictement) un seul sous-ensemble positif : {200} ;
• *B* en contient quatre : {1}, {2}, {1, 2} et {-1, 2} ;
• *C* en contient sept : {4}, trois fois {4, -1} et trois fois {4, -1, -1} ;
• *D* en contient sept également : trois fois {1}, trois fois {1, 1}, et {1, 1, 1}.

Cet article est une version vulgarisée d'un article spécialisé publié tout récemment. Le thème central en est :

Combien de parties positives contient un ensemble positif ?

Suppose we have 100 real numbers such that the sum of them is positive. How many pairs can we form out of these numbers whose sum will still be positive ?

Of course all of these 100 numbers can be positive and so all the pairs will give a positive sum, so it is meaningful only to ask the minimum possible number of such pairs. Everybody would have the feeling that at least some of the pairs will give a positive sum (why ?) but how many ?

Another natural choice for these numbers is to pick one = 100 and the remaining 99 = -1, in which case only the pairs containing the number 100 will be positive (we will call a set of numbers *positive* if their sum is positive) and so we will have only 99 positive pairs.

Is this number the minimum or is it still possible to have a smaller number of positive pairs ?

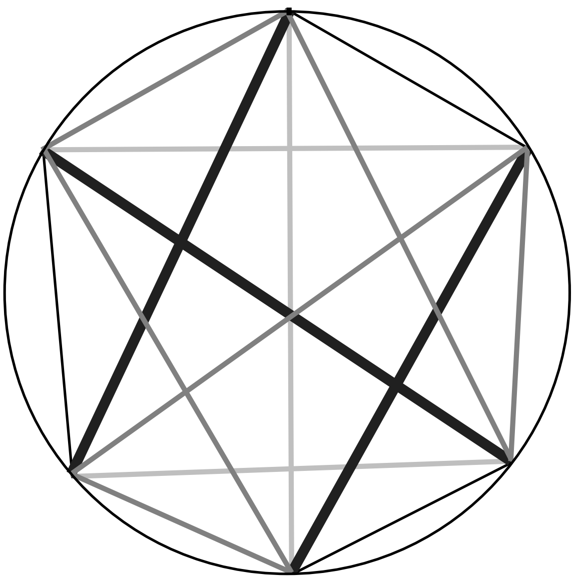
If we partition (i.e. group into sets of size 2) the 100 numbers into pairs then definitely at least one of these pairs will be positive, since otherwise the sum of all the 100 numbers being equal to the sum of the 50 sums of the numbers in each of the pairs would be negative, a contradiction.

Let us try to form these partitions such that every pair of the given numbers is exactly in one of them : we have altogether 100·99/2 pairs (why ?) and each partition of the 100 numbers contain exactly 50 pairs, so we would need to use 99 partition (luckily the same 99).

But is it possible to group the 100·99/2 pairs of these 100 numbers into 99 groups such that each group will form a partition of the 100 numbers ?

The answer is yes, as in general we can do such a way for any even number of numbers.

The following picture shows, e.g., how to group the pairs of 6 given numbers into partitions of the 6 numbers. The points on the circle represent the 6 given numbers and the segments in between the points represent the pairs, similarly drawn segments pairs belonging to the same groups.



(Try to do the same for 8, 10 or even 100 numbers !)

Thus now we know that in case of 100 numbers of positive sum at least 99 pairs of these numbers will be positive again.

Of course we have not used the fact that we have exactly 100 numbers, only that this number was even : it was necessary to have even numbers to be able to partite them into pairs !

Let us repeat the same question with 33 numbers and triplets, that is how many positive triplets do we have (at least) from a set of 33 numbers which have positive sum ?

Again, we may have all the 33 numbers and then all triplets positive or we may have as choosen 33 numbers one copy of 33 and 32 copies of -1 again.

How many positive triplets do we have in this second case ?

Naturally, these triplets must contain the number 33 and two of the many -1's ; there are 32·31/2 ways to choose a pair of -1's, so we will have exactly this many positive triplets. How do we prove again that this is the possible minimum one.

It was a good idea to use the fact that every partitioning of the set of numbers into subsets of the given size (pair earlier or triplet now) must contain at least one positive subset. But can we partition now the triplets of the 33 numbers into groups each containing each of the 33 numbers exactly once ?

The answer is yes but it would be too difficult to prove …

> Si n est multiple de k, il est possible en général de partitionner les $\binom{n}{k}$ parties à k éléments d'un ensemble à n éléments en groupes, chacun formant une partition de E. Cela résulte d'un difficile et remarquable théorème montré en 1979 par Baranyai.

… so let us look for another proof.

In combinatoriccs one of the early standard methods is the so-called *double counting,* which will be demonstrated here : for a given set of 33 real numbers a_1, a_2, … a_{33} with positive sum let T denote the set of the positive triplets and $\mathbf{P}$ denote the set of all possible partitions of the 33 numbers into triplets :

$$\mathbf{P} = \{(a_1\, a_2\, a_3)\ldots(a_{31}\, a_{32}\, a_{33}),\ (a_1\, a_2\, a_4)\,(a_3\, a_5\, a_6)\ldots(a_{31}\, a_{32}\, a_{33})\ldots\}$$

[NDLR : les partitions sont ici listées comme dans un dictionnaire à deux étages : les *triples* d'une partition sont rangés par ordre “alphabétique”, les nombres correspondant aux lettres ; les partitions sont rangées également par ordre alphabétique, les triplets correspondant cette fois-ci aux lettres.] [NDLC : Les mathématiciens parleraient ici d'“ordre lexicographique produit”, mais la parole est à Miklós Dezsö (au fait c'est ó ou c'est ö le premier dans l'ordre lexicographique ?) :]

… then the size of $\mathbf{P}$ is

$$\left|\mathbf{P}\right| = \frac{\binom{33}{3}\cdot\binom{30}{3}\cdots\binom{6}{3}\cdot\binom{3}{3}}{11!}$$

or

$$\left|\mathbf{P}\right| = \frac{\frac{33\cdot 32\cdot 31}{6}\cdot\frac{30\cdot 29\cdot 28}{6}\cdots\frac{6\cdot 5\cdot 4}{6}\cdot\frac{3\cdot 2\cdot 1}{6}}{11!}$$

because the first triplet can be chosen 33·32·31/6 different ways (the first element of it can be any of the 33 numbers, the next any of the remaining 32 numbers while the last any of the still remaining 31 numbers, but then we choose a triplet at all the possible 6 different ordering of its three element), the second similarly 30·29·28/6 ways, etc., but then all the partitions were chosen 11 ! = 11·10·9·8·7·6·5·4·3·2·1 different times at the 11 ! possible ordering of the 11 triplets contained by it.

Now count in two different ways the number of the following pairs :

$$\{(\mathrm{T}, \mathrm{P})\ :\ \mathrm{T} \text{ is from } \mathbf{T},\ \mathrm{P} \text{ is from } \mathbf{P} \text{ and T is a triplet in P}\}$$

One way to count is to see how many partitions P contains a certain triplet T. This number will be certainly equal to the number of partitions of 30 (=33-3) numbers into triplets which is similarly to size of $\mathbf{P}$ is equal to

$$\frac{\binom{30}{3}\cdot\binom{27}{3}\cdots\binom{6}{3}\cdot\binom{3}{3}}{10!}$$

or

$$\frac{\frac{30\cdot 29\cdot 28}{6}\cdot\frac{27\cdot 26\cdot 25}{6}\cdots\frac{6\cdot 5\cdot 4}{6}\cdot\frac{3\cdot 2\cdot 1}{6}}{10!}$$

and so the number of pairs is equal to

$$\left|\mathbf{T}\right|\cdot\frac{\frac{30\cdot 29\cdot 28}{6}\cdot\frac{27\cdot 26\cdot 25}{6}\cdots\frac{6\cdot 5\cdot 4}{6}\cdot\frac{3\cdot 2\cdot 1}{6}}{10!}$$

On the other hand, we know that every partition of the 33 numbers into triplets must contain at least one positive triplet, so the number of the above pairs is at least the size of $\mathbf{P}$, and so we have that

$$\left|\mathbf{T}\right|\cdot\frac{\frac{30\cdot 29\cdot 28}{6}\cdot\frac{27\cdot 26\cdot 25}{6}\cdots\frac{6\cdot 5\cdot 4}{6}\cdot\frac{3\cdot 2\cdot 1}{6}}{10!} \geq \left|\mathbf{P}\right| = \frac{\frac{33\cdot 32\cdot 31}{6}\cdot\frac{30\cdot 29\cdot 28}{6}\cdots\frac{6\cdot 5\cdot 4}{6}\cdot\frac{3\cdot 2\cdot 1}{6}}{11!}$$

Or

$$\left|\mathbf{T}\right| \geq \frac{33\cdot 32\cdot 31}{6\cdot 11} = \frac{32\cdot 31}{2}$$

giving that in any given case of 33 numbers with positive sum there must be at least 32·31/2 positive triplets formed by these numbers.

The previous statements and proofs can now be generalized in the following theorem :

THEOREM If *k* and *n* are two positive integers and *k* is a divisor of *n*, and there are *n* real numbers given with a positive sum the minimum number of positive *k* -subsets (subsets of size *k*) of the given numbers will be given, for example, by the following *n* numbers :

$$n, -1, -1, \ldots, -1, -1$$

Of course for the statement of the theorem we do not need that k is a divisor of n, only the proofs given so far require this fact.

Is then the generalization of the above statement true for an arbitrary pair of two positive integers $k < n$?

The answer is ***no***, as the following examples will show, where on the right hand side always the distribution of the preceeding examples is shown while on the left hand side another distibution of the n real numbers with positive sum as well but with fewer number positive triplets (in these examples k is always 3).

EXAMPLE 1

$k = 3$, $n = 5$

(1, 1, 1, 1, -3)	(5, -1, -1, -1, -1)
4 positive triplets	6 positive triplets

EXAMPLE 2

$k = 3$, $n = 8$

(2, 2, 2, 2, 2, 2,-5,-5)	(8,-1,-1,-1,-1,-1,-1,-1)
20 positive triplets	21 positive triplets

EXAMPLE 3

$k = 3$, $n = 10$

(4,4,4,4,4,4,4,-9,-9,-9)	(10,-1,-1,-1,-1,-1,-1,-1,-1,-1)
35 positive triplets	36 positive triplets

Of course these examples are not the only ones.

Using them, try to give general examples of distribution of n real numbers with positive sum such that the number of positive k -subsets is always less then in the case of the distribution $\{n, -1, -1, \dots -1\}$ for the choices of $n = k + 2$, $2k + 1$ and $3k + 1$.

Having these negative results as well one could easily form the following conjecture :

CONJECTURE The statement of the previous theorem is true for a pair k and n of positive integers if and only if k is a divisor of n.

… which turns out to be ***false*** again ! ! !

[NDLC : attention, dans ce qui suit, k ‘s est le pluriel de k.] The situation is even more complicated : even for the cases when k does not divide n we have the above counterexamples only for relatively small k ‘s (compared to n) and so the following general theorem is true :

THEOREM’ If k and n are positive integers and either k is a divisor of n or $n > c(k)$, a constant depending only on k, then for any given n real numbers with a positive sum the minimum number of positive k-subsets is given by the example when the n numbers are of the form of

$$\{n, -1, -1, \dots -1\}.$$

Of course a few questions still remain open :

1) For the remaining cases, that is those which are not handled by THEOREM’ what are the minimum numbers of the positive k-subsets ?

2) What are at all the solved cases, that is what is the size of $c(k)$?

Unfortunately we do not know the answers to most of the questions. However, there is one more phenomenon worth looking because it is quite common when doing research : the few known values of $c(k)$ are very small, but we know only a very big general bound for $c(k)$:

$$c(2) = 5,\ c(3) = 10,$$

while in general we only know that

$$c(k) < (2k)^{(k+1)}.$$