$a^2 + b^2 = c^2$


enseignant : Mr Gert Schomacker

Pythagorean triangles
i.e. the sides are integers.

We have been working with pythagorean triangles. During the investigation of the subject we also dealt with Archemedean triangles. Furthermore we had a contest. All this you may read more about is the following.

$a^2 + b^2 = (b + 1)^2$

The purpose is to determine the length of the sides in a rectangular triangle, in which the hypotenuse is 1 bigger than the longest side.

\[
\begin{align*}
a^2 + b^2 &= (b + 1)^2 \\
a^2 + b^2 &= b^2 + 2b + 1 \\
a^2 &= 2b + 1 \\
b &= (a^2 - 1)/2
\end{align*}
\]

Then $c = b + 1 \rightarrow c = (a^2 + 1)/2$

We can add up these results:

\[(a, b, c) = (a, (a^2 - 1)/2, (a^2 + 1)/2), \quad a = 3, 5, 7, 9, \ldots\]

Here are some examples:

\[
\begin{align*}
3^2 + 4^2 &= 5^2 \\
5^2 + 12^2 &= 13^2 \\
7^2 + 24^2 &= 25^2
\end{align*}
\]

$a$ cannot be an even number. If you try to put in an even number into the formulae for $b$ and $c$, the result will not be an integer.

$a^2 + b^2 = (b + 2)^2$

This formula is used to find integer sides on pythagorean triangles. You can find the sides of the triangles with a hypotenuse $c$ that is 2 bigger than $b$.

From Pythagoras:

\[
\begin{align*}
a^2 + b^2 &= c^2 \\
a^2 + b^2 &= (b+2)^2 \\
a^2 + b^2 &= b^2 + 4b \\
a^2 + b^2 &= b^2 + 4b + 4 \\
b &= (a^2 - 4)/4
\end{align*}
\]
Then
\[ c = b + 2 \]
\[ c = (a^2 + 4)/4 \]

We came to these results:
\[(a, b, c) = (a, (a^2 - 4)/4, (a^2 + 4)/4)\]

Examples:
\[
\begin{align*}
4^2 + 3^2 & = 5^2 \\
8^2 + 15^2 & = 17^2 \\
12^2 + 35^2 & = 37^2
\end{align*}
\]
a must be divisible by 4 (a = 4, 8, 12, 16, 20 ...). If a is odd, b and c will not become integers. If a is divisible by 2 but not by 4 the triple (a, b, c) have a common factor.

For instance a = 6 gives the triple (6, 8, 10) ~ (3, 4, 5).

**Our purpose is to find a systematic way to determine all primitive pythagorian triples**

We want to prove that:
\[(a, b, c)\] can be written as
\[(2pq, p^2 - q^2, p^2 + q^2),\]
when the conditions for p and q are:
• not both odd
• p > q
• no common factors

We assume that a is even and b, c are odd.

**PYTHAGORAS**

\[ a^2 = c^2 - b^2 \]
\[ a^2 = (c + b)(c - b) \]

divided by two:
\[
\begin{align*}
(a/2)^2 & = ((c + b)/2)((c - b)/2) \\
(c + b)/2 & = p^2
\end{align*}
\]
p and q have to be squares so that \((a/2)^2\) can be a square

\[ (c - b)/2 = q^2 \]
\[ p^2 + q^2 = ((c + b)/2) + ((c - b)/2) = c \]
\[ p^2 - q^2 = ((c + b)/2) - ((c - b)/2) = b \]
\[ (a/2)^2 = a^2/4 \]
\[ c^2/4 = p^2q^2 \]
\[ a^2 = 4 \cdot (p^2 \cdot q^2) \]

squeareroot
\[ a = 2 (p \cdot q) \]

**Archimedean triangles**

A triangle with one angle \(\pi/3\) (60°) and with integer sides is called an archimedean triangle. To find a general formula for these triangles, our startingpoint is the cosine-relations.

Cosine:
\[ a^2 = b^2 + c^2 - 2bc \cdot \cos(60°) \]
where \(\cos(60°) = 1/2\)

**Formula**:

\[
\begin{align*}
 a^2 & = b^2 + c^2 - bc \\
 4a^2 & = 4b^2 + 4c^2 - 4bc \\
 4a^2 - 4c^2 - b^2 + 4bc & = 3b^2 \\
 (4a^2) - (4c^2 - 4bc + b^2) & = 3b^2 \\
 (2a)^2 - (2c - b)^2 & = 3b^2 \\
 (2a + 2c - b)(2a - 2c + b) & = 3b^2 \\
 3p^2q^2 & = 3b^2 \\
 a & = (3p^2 + q^2)/4 \\
 b & = p \cdot q \\
 c & = (3p^2 - q^2 + 2pq)/4
\end{align*}
\]

We found some archimedean triangles from the formula and from this we can conclude that there is no system what so ever.
Examples of archimedean triangles:

\[
\begin{align*}
  p &= q = 1 & (a, b, c) &= (1, 1, 1) \\
  p &= 3, q = 1 & (a, b, c) &= (7, 3, 8) \\
  p &= 3, q = 5 & (a, b, c) &= (13, 15, 8) \\
  p &= 5, q = 1 & (a, b, c) &= (19, 5, 21) \\
  p &= 5, q = 3 & (a, b, c) &= (21, 15, 24)
\end{align*}
\]

Contest:

We made a contest, where you were to find \(c\) and \(b\), when \(a\) equals 1994. \(c\) and \(b\) were to be integers. The prize winner of a FG-sweat-shirt was Siek-Hor Lim.

\[
\begin{align*}
  \left(\frac{a}{2}\right)^2 &= \left(\frac{c + b}{2}\right)\left(\frac{c - b}{2}\right) \\
  (1994/2)^2 &= 997^2 \\
  \frac{c + b}{2} &= 997^2 \\
  \frac{c - b}{2} &= 1 \\
  \Rightarrow c - 1 &= 997^2 = 994009 \\
  c - b &= 2 \\
  994009 + 1 &= 994010 \\
  b + 2 &= c \\
  994009 - 1 &= 994008 \\
  \text{(The difference is two)} \\
  (a, b, c) &= (1994, 994008, 994010)
\end{align*}
\]

You are to find the sides \(b\) and \(c\) satisfying the Pythagorean theorem \(a^2 + b^2 = c^2\) in which \(a = 1994\)

The numbers have to be integers.

*Contest*

If you have any solutions, please give them to one of us.

The winner will get a prize!

We will announce the winner + give the procedure and solution Monday about 2 pm.